Schemes 2018 Excersize 1

Question 1. Recall that, for a ring R, the **Zariski topology** on $spec(R)$ is defined by the property that for every ideal $I \subseteq R$, the set of prime ideal containing I is closed. Prove that the Zariski topology is indeed a topology.

Question 2. Recall that a map of rings $\phi : R \to S$ induces a morphism on the spectra $\phi^{-1} : spec(S) \to spec(R)$. Prove that it is continuous in Zariski topology.

Question 3. Let R be a commutative ring. Prove that the following are equivalent:

- $R = R_1 \oplus R_2$ for non-zero rings R_1 and R_2.
- $spec(R)$ is not connected.
- R contains an **idempotent**: an element r such that $r^2 = r$ but $r \neq 1$.

Question 4. Let $R = \mathbb{Z}$, and $S = \mathbb{Z}[i]$. Let $\phi : \mathbb{Z} \to \mathbb{Z}[i]$ be the inclusion. Show that ϕ^{-1} is surjective, and show that the number of elements in the pre-image of a prime $(p) \subseteq \mathbb{Z}$ is 1 or 2, depending only on the residue of p mod 4.

We shall denote by $spec_m(R)$ the maximal spectrum of R, i.e. the topological space of all maximal ideals with the induced topology from Zariski topology on the spectrum of R.

Question 5 (*)**.** Let X be a compact Hausdorff topological space. Let $C(X)$ denote the ring of continuous, $\mathbb{C}$ valued functions on X. Let $\mathfrak{m}$ be a maximal ideal in $C(X)$. Show that $C(X)/\mathfrak{m} \cong \mathbb{C}$ and that in fact there exists $x \in X$ such that $\mathfrak{m} = \{f \in C(X) : f(x) = 0\}$. Deduce that $spec_m(C(X)) = X$ as sets. Show that in fact the two are homeomorphic as topological spaces, where $spec_m(C(X))$ is endowed with the Zariski topology, restricted to the maximal spectrum.